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ON RIGOROUS STABILITY CONDITIONS FOR DYNAMIC QUASI-BIFURCATIONS

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Abstract—In this present research, the dynamic quasi-bifurcation stability theory is studied in detail. After some basic concepts of stability of structural dynamical systems have been introduced, sufficient conditions for instability and necessary conditions for stability of discrete systems are derived. These conditions are expressed in terms of the properties of structural mass, tangential damping and stiffness matrices. Thus the paper allows an easy computation of an upper bound of critical time, in which a structural response way departs into instability.

1. INTRODUCTION

The concept of "quasi-bifurcation" in dynamics was first introduced by Lee (1977). It may be stated as follows: in analyzing the stability of an unperturbed response of a dynamic system, all adjacent perturbed motions are to be investigated. When the unperturbed motion is stable, all perturbed motions will remain in the neighborhood of the original response; when unstable, at least one perturbed motion departs from the neighborhood. This means certain infinitesimal disturbances may amplify with time, if the system is unstable.

Based on this idea, a criterion of dynamic stability was established in terms of the properties of an effective force field (Lee, 1977, 1981). Such a criterion was extended by Kleiber *et al.* (1986) to discrete dynamical systems. A direct objective of developing the dynamic quasi-bifurcation stability (DQBS) criterion was to determine the behavior of a response which is stable in a limited time interval $0 < t < t_{cr}$, but unstable for $t > t_{cr}$. Kleiber's attempt was to determine the so-called critical time t_{cr} by the DQBS criterion.

As the present study will demonstrate, there exist some weak prerequisites in the derivation of the DQBS criterion in both papers of Lee (1977, 1981) and Kleiber *et al.* (1986). In fact, a rigorous derivation of the DQBS criterion only offers a sufficient condition for instability and a necessary condition for stability, as will be shown.

2. PERTURBED EQUATIONS OF MOTION

Let us consider the response of an arbitrary discrete system with n degrees of freedom. The nodal displacements and loads are denoted by:

$$\mathbf{V} = [V_1, V_2, \ldots, V_n]^T, \quad \mathbf{P} = [P_1, P_2, \ldots, P_n]^T. \tag{1}$$

Any load-deflection path $\mathbf{V}(\mathbf{P}, t)$ shall be described by the nonlinear equations of motion

$$\mathbf{M} \cdot \ddot{\mathbf{V}} + \mathbf{G}(\mathbf{V}, \dot{\mathbf{V}}, t) = \mathbf{P}(t), \quad (\dot{\ldots}) = \mathrm{d}\ldots/\mathrm{d}t \tag{2}$$

with the global mass matrix $\mathbf{M}$, nodal velocities $\dot{\mathbf{V}}$ and accelerations $\ddot{\mathbf{V}}$, the matrix function $\mathbf{G}$ of internal nodal forces due to viscous damping as well as elasto-plastic restoring mechanisms (Krätzig, 1990).

It is well known that the nonlinear response $\mathbf{V}(\mathbf{P}, t)$ of (2) may exhibit various instability phenomena, e.g. bifurcating, diverging or snapping. In order to construct incremental–

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iterative procedures for the detection of those responses including their instabilities we decompose (1)

$$\mathbf{V} = \bar{\mathbf{V}} + \overset{+}{\mathbf{V}}, \quad \mathbf{P} = \bar{\mathbf{P}} + \overset{+}{\mathbf{P}} \tag{3}$$

into the fundamental state $\bar{\mathbf{V}}$, $\bar{\mathbf{P}}$ and infinitesimal perturbations $\overset{+}{\mathbf{V}}$, $\overset{+}{\mathbf{P}}$ of the adjacent state. Analogous decompositions follow for $\dot{\mathbf{V}}$, $\ddot{\mathbf{V}}$. Forming the first variation of (2) with respect to the fundamental state, we receive

$$\mathbf{M} \cdot \ddot{\overset{+}{\mathbf{V}}} + \mathbf{C}_T \cdot \dot{\overset{+}{\mathbf{V}}} + \mathbf{K}_T \cdot \overset{+}{\mathbf{V}} = \overset{+}{\mathbf{P}} \tag{4}$$

in which the following abbreviations were introduced:

$$\mathbf{C}_T = \left.\frac{\partial \mathbf{G}}{\partial \dot{\mathbf{V}}}\right|_{\bar{\mathbf{V}}} = \text{tangential damping matrix},$$

$$\mathbf{K}_T = \left.\frac{\partial \mathbf{G}}{\partial \mathbf{V}}\right|_{\bar{\mathbf{V}}} = \text{tangential stiffness matrix}.$$

In the concept of structural stability, eqns (2) and (4) are denoted as unperturbed and perturbed (or tangential) equations of motion, respectively. If the decompositions of (3) are accomplished at time t_0, the perturbed equations of motion (4) are generally used to determine the stability of the unperturbed response (2) at time t_0, possibly in the sense of Liapunov (1893). When the original response $\bar{\mathbf{V}}$ is stable (asymptotically stable), all perturbed motions have to remain in a defined neighborhood of $\bar{\mathbf{V}}$ (and damp out for $t \to \infty$). When unstable, at least one perturbed motion will depart from the neighborhood, and the originally infinitesimal perturbations will amplify with time. Physically, such a divergence is called a quasi-bifurcation phenomenon (Lee, 1977).

3. INSTABILITY ASSESSMENT FOR GENERAL DYNAMICAL SYSTEMS

Before evaluating directly the stability of the unperturbed motion, we first investigate the following two integral identities:

$$\int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau}[\dot{\overset{+}{\mathbf{V}}}{}^T(\tau) \cdot \overset{+}{\mathbf{V}}(\tau)]\,\mathrm{d}t = \int_{t_0}^{t} [\ddot{\overset{+}{\mathbf{V}}}{}^T(\tau) \cdot \overset{+}{\mathbf{V}}(\tau) + \dot{\overset{+}{\mathbf{V}}}{}^T(\tau) \cdot \dot{\overset{+}{\mathbf{V}}}(\tau)]\,\mathrm{d}\tau,$$

$$\int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau}[\ddot{\overset{+}{\mathbf{V}}}{}^T(\tau) \cdot \dot{\overset{+}{\mathbf{V}}}(\tau)]\,\mathrm{d}t = \int_{t_0}^{t} [\dddot{\overset{+}{\mathbf{V}}}{}^T(\tau) \cdot \dot{\overset{+}{\mathbf{V}}}(\tau) + \ddot{\overset{+}{\mathbf{V}}}{}^T(\tau) \cdot \ddot{\overset{+}{\mathbf{V}}}(\tau)]\,\mathrm{d}\tau. \tag{5}$$

By integrating twice the terms on the left-hand side of (5), we obtain

$$\tfrac{1}{2}\overset{+}{\mathbf{V}}{}^T(t) \cdot \overset{+}{\mathbf{V}}(t) = \tfrac{1}{2}\overset{+}{\mathbf{V}}{}^T(t_0) \cdot \overset{+}{\mathbf{V}}(t_0) + \int_{t_0}^{t} [\dot{\overset{+}{\mathbf{V}}}{}^T(t_0) \cdot \overset{+}{\mathbf{V}}(t_0)]\,\mathrm{d}t$$
$$+ \int_{t_0}^{t} \int_{t_0}^{\tau} [\ddot{\overset{+}{\mathbf{V}}}{}^T(\tau) \cdot \overset{+}{\mathbf{V}}(\tau) + \dot{\overset{+}{\mathbf{V}}}{}^T(\tau) \cdot \dot{\overset{+}{\mathbf{V}}}(\tau)]\,\mathrm{d}t\,\mathrm{d}\tau,$$

$$\tfrac{1}{2}\dot{\overset{+}{\mathbf{V}}}{}^{T}(t)\cdot\dot{\overset{+}{\mathbf{V}}}(t) = \tfrac{1}{2}\dot{\overset{+}{\mathbf{V}}}{}^{T}(t_0)\cdot\dot{\overset{+}{\mathbf{V}}}(t_0) + \int_{t_0}^{t}[\ddot{\overset{+}{\mathbf{V}}}{}^{T}(t_0)\cdot\dot{\overset{+}{\mathbf{V}}}(t_0)]\,\mathrm{d}t + \int_{t_0}^{t}\int_{t_0}^{\tau}[\dddot{\overset{+}{\mathbf{V}}}{}^{T}(\tau)\cdot\dot{\overset{+}{\mathbf{V}}}(\tau)+\ddot{\overset{+}{\mathbf{V}}}{}^{T}(\tau)\cdot\ddot{\overset{+}{\mathbf{V}}}(\tau)]\,\mathrm{d}t\,\mathrm{d}\tau, \quad (6)$$

where $\overset{+}{\mathbf{V}}(t_0)$ and $\dot{\overset{+}{\mathbf{V}}}(t_0)$ are the initial conditions of $\overset{+}{\mathbf{V}}(t)$ at time t_0. Let

$$Y(t) = \tfrac{1}{2}\overset{+}{\mathbf{V}}{}^{T}(t)\cdot\overset{+}{\mathbf{V}}(t), \quad Z(t) = \tfrac{1}{2}\dot{\overset{+}{\mathbf{V}}}{}^{T}(t)\cdot\dot{\overset{+}{\mathbf{V}}}(t). \quad (7)$$

Then, we have

$$\dot{Y}(t) = \dot{\overset{+}{\mathbf{V}}}{}^{T}(t_0)\cdot\overset{+}{\mathbf{V}}(t_0) + \int_{t_0}^{t}[\ddot{\overset{+}{\mathbf{V}}}{}^{T}(t)\cdot\overset{+}{\mathbf{V}}(t)+\dot{\overset{+}{\mathbf{V}}}{}^{T}(t)\cdot\dot{\overset{+}{\mathbf{V}}}(t)]\,\mathrm{d}t,$$

$$\dot{Z}(t) = \ddot{\overset{+}{\mathbf{V}}}{}^{T}(t_0)\cdot\dot{\overset{+}{\mathbf{V}}}(t_0) + \int_{t_0}^{t}[\dddot{\overset{+}{\mathbf{V}}}{}^{T}(t)\cdot\dot{\overset{+}{\mathbf{V}}}(t)+\ddot{\overset{+}{\mathbf{V}}}{}^{T}(t)\cdot\ddot{\overset{+}{\mathbf{V}}}(t)]\,\mathrm{d}t. \quad (8)$$

The initial conditions for the new functions $Y(t)$ and $Z(t)$ are

$$Y(t_0) = \tfrac{1}{2}\overset{+}{\mathbf{V}}{}^{T}(t_0)\cdot\overset{+}{\mathbf{V}}(t_0) \geqslant 0, \quad \dot{Y}(t_0) = \dot{\overset{+}{\mathbf{V}}}{}^{T}(t_0)\cdot\overset{+}{\mathbf{V}}(t_0),$$

$$Z(t_0) = \tfrac{1}{2}\dot{\overset{+}{\mathbf{V}}}{}^{T}(t_0)\cdot\dot{\overset{+}{\mathbf{V}}}(t_0) \geqslant 0, \quad \dot{Z}(t_0) = \ddot{\overset{+}{\mathbf{V}}}{}^{T}(t_0)\cdot\dot{\overset{+}{\mathbf{V}}}(t_0). \quad (9)$$

Mathematically, $Y(t)$ and $Z(t)$ characterize the squares of the norms of displacement and velocity vectors in the phase plane. According to Liapunov's general definitions (1893) we find the original motion stable, if, after a set of sufficiently small initial disturbances, $\overset{+}{\mathbf{V}}(t_0)$, $\dot{\overset{+}{\mathbf{V}}}(t_0)$, all perturbed motions remain small. Vice versa, if a set of small initial disturbances results in only one perturbed motion which amplifies with time, the unperturbed motion is unstable. Obviously, if the solution $\overset{+}{\mathbf{V}}(t)$ is known, not only for one particular set of initial values, but also for all initial values in the neighborhood of the original motion, it is easy to determine whether the criterion for Liapunov stability is satisfied or not. In general, however, the solution is evaluated for a particular set of initial values, but not known for all initial values. Therefore, it is of interest to develop other more direct criteria which enable us to determine the stability of responses without the evaluation of all possible solutions for all initial values.

On inspection of expressions (7) and (8), we observe two noteworthy features. First, we consider the inequality

$$\int_{t_0}^{t}[\ddot{\overset{+}{\mathbf{V}}}{}^{T}(t)\cdot\overset{+}{\mathbf{V}}(t)+\dot{\overset{+}{\mathbf{V}}}{}^{T}(t)\cdot\dot{\overset{+}{\mathbf{V}}}(t)]\,\mathrm{d}t \geqslant 0. \quad (10)$$

In this case we can find at least one initial condition $\dot{Y}(t_0) > 0$ with $\dot{Y}(t) > 0$ for $t \geqslant t_0$. Thus $Y(t)$ will increase monotonously with time. This means the infinitesimal disturbances may amplify with time and thus the unperturbed motion will certainly be unstable. On the other hand, if

$$\int_{t_0}^{t}[\ddot{\overset{+}{\mathbf{V}}}{}^{T}(t)\cdot\overset{+}{\mathbf{V}}(t)+\dot{\overset{+}{\mathbf{V}}}{}^{T}(t)\cdot\dot{\overset{+}{\mathbf{V}}}(t)]\,\mathrm{d}t < 0, \quad (11)$$

$\dot{Y}(t)$ may be either positive or negative since the initial condition, $\dot{Y}(t_0)$, is at one's own choice. Thus, the boundedness of $Y(t)$ is unknown and the stability is also undetermined. Second, we consider another inequality:

$$\ddot{\overset{+}{\mathbf{V}}}{}^{T}(t_0)\cdot\dot{\overset{+}{\mathbf{V}}}(t_0) > 0. \tag{12}$$

In this case, $Z(t)$ is an increasing function at time t_0. However, if

$$\ddot{\overset{+}{\mathbf{V}}}{}^{T}(t_0)\cdot\dot{\overset{+}{\mathbf{V}}}(t_0) \leqslant 0, \tag{13}$$

$Z(t)$ is a nonincreasing function.

These properties mentioned above demonstrate that inequalities (10) and (12) may be employed as the sufficient conditions for instability, while inequalities (11) and (13) form a necessary condition for stability, respectively, for general dynamical responses.

It is interesting to examine briefly the nature of inequality (10). The first term represents the projection of the acceleration vector $\ddot{\overset{+}{\mathbf{V}}}(t)$, at any neighborhood position, on $\overset{+}{\mathbf{V}}(t)$. The second term presents the square of the norm of the velocity vector $\dot{\overset{+}{\mathbf{V}}}(t)$. Because the latter is non-negative, inequality (10) can be abbreviated as

$$\int_{t_0}^{t} [\ddot{\overset{+}{\mathbf{V}}}{}^{T}(t)\cdot\overset{+}{\mathbf{V}}(t)]\,\mathrm{d}t \geqslant 0. \tag{14}$$

It is important to note that if $\overset{+}{\mathbf{V}}(t)$ satisfies inequality (14), it certainly will satisfy inequality (10). However, $\overset{+}{\mathbf{V}}(t)$ satisfying (10) may not satisfy (14). This proves that inequality (10) may give a more exact assessment for instability than inequality (14). Thus, inequality (14) is only a sufficient condition for instability.

Referring to the papers of Lee (1977, 1981) and Kleiber *et al.* (1986), inequality (14) was employed as a condition to determine a so-called critical time till instability. The analysis given here clearly demonstrates that inequality (14) is only a sufficient condition for instability rather than a critical condition. In fact, critical times to be determined by either (14) or (10), at the utmost can be regarded as an upper limit up to a possible onset of instability.

4. INSTABILITY CONDITIONS IN STRUCTURAL DYNAMICS

Now let us change to stability problems in the field of structural dynamics. Consider the applied perturbations being produced by the initial conditions. The perturbed equations of motion (4) thus will remain a homogeneous form, i.e.

$$\mathbf{M}\cdot\ddot{\overset{+}{\mathbf{V}}}+\mathbf{C}_{\mathrm{T}}\cdot\dot{\overset{+}{\mathbf{V}}}+\mathbf{K}_{\mathrm{T}}\cdot\overset{+}{\mathbf{V}} = \mathbf{0}. \tag{15}$$

Because the mass matrix $\mathbf{M}$, in general, is positive definite, eqn (15) may be rewritten as follows:

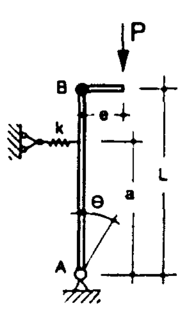

Fig. 1. Geometry and sign of simple 1-D model.

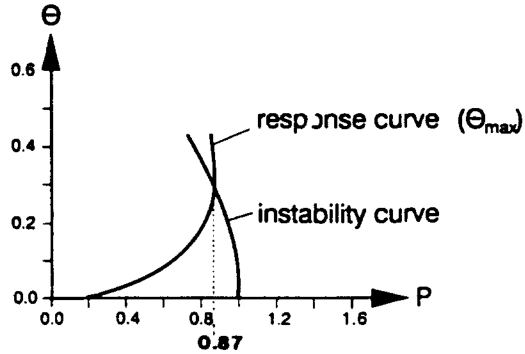

Fig. 2. Response and instability curves.

$$\ddot{\overset{+}{\mathbf{V}}} = -\mathbf{M}^{-1}\cdot\mathbf{C}_T\cdot\dot{\overset{+}{\mathbf{V}}} - \mathbf{M}^{-1}\cdot\mathbf{K}_T\cdot\overset{+}{\mathbf{V}}. \tag{16}$$

Substituting (16) into inequalities (11) and (13) leads to

$$\int_{t_0}^{t}[\dot{\overset{+}{\mathbf{V}}}^T\cdot\dot{\overset{+}{\mathbf{V}}} - \dot{\overset{+}{\mathbf{V}}}^T\cdot\mathbf{M}^{-1}\cdot\mathbf{C}_T\cdot\dot{\overset{+}{\mathbf{V}}} - \dot{\overset{+}{\mathbf{V}}}^T\cdot\mathbf{M}^{-1}\cdot\mathbf{K}_T\cdot\overset{+}{\mathbf{V}}]\,\mathrm{d}t < 0,$$

$$\dot{\overset{+}{\mathbf{V}}}^T(t_0)\cdot[-\mathbf{M}^{-1}\cdot\mathbf{C}_T\cdot\dot{\overset{+}{\mathbf{V}}}(t_0) - \mathbf{M}^{-1}\cdot\mathbf{K}_T\cdot\overset{+}{\mathbf{V}}(t_0)] \leqslant 0. \tag{17}$$

The first inequality of (17) may be further expressed by a matrix form

$$\int_{t_0}^{t}[\dot{\overset{+}{\mathbf{V}}}^T\overset{+}{\mathbf{V}}^T]\cdot\begin{bmatrix}\mathbf{I} & 0\\ -\mathbf{M}^{-1}\cdot\mathbf{C}_T & -\mathbf{M}^{-1}\cdot\mathbf{K}_T\end{bmatrix}\cdot\begin{bmatrix}\dot{\overset{+}{\mathbf{V}}}\\ \overset{+}{\mathbf{V}}\end{bmatrix}\mathrm{d}t < 0. \tag{18}$$

Note that the initial perturbations are arbitrarily given. The following condition thus is necessary so that inequalities (17) are satisfied:

$$(\mathbf{M}^{-1}\cdot\mathbf{C}_T) \geqslant 0 \quad \text{and} \quad (\mathbf{M}^{-1}\cdot\mathbf{K}_T) > 0. \tag{19}$$

Inequality (19) represents a necessary condition for stability for structural dynamical

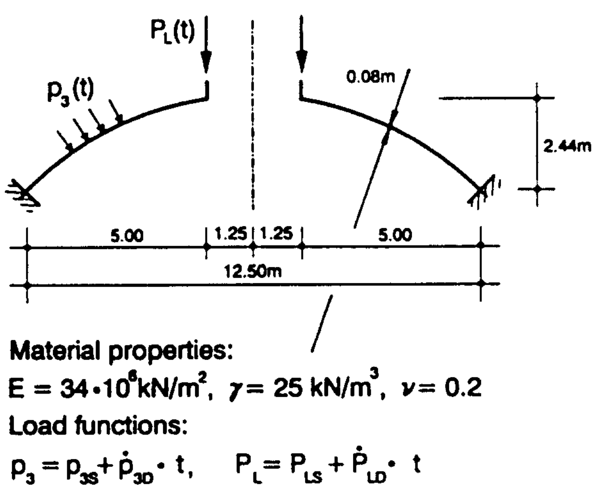

Fig. 3. Geometry and parameters of isotropic spherical cap (units: m).

systems. Similarly, sufficient conditions for instability can also be obtained from inequalities (10) and (12):

$$(\mathbf{M}^{-1}\cdot\mathbf{C}_T) \not\geqslant 0 \quad \text{or} \quad (\mathbf{M}^{-1}\cdot\mathbf{K}_T) \not> 0. \tag{20}$$

Note that both matrices $(\mathbf{M}^{-1}\cdot\mathbf{C}_T)$ and $(\mathbf{M}^{-1}\cdot\mathbf{K}_T)$ are unsymmetric. Thus the assessment for instability will concern the problem of examination of an unsymmetric matrix whether it is positive definite or positive semi-definite. Mathematically it is defined that the matrix is positive definite or positive semi-definite if and only if all eigenvalues of this matrix are positive or non-negative reals, otherwise it is neither positive definite nor positive semi-definite. However, if matrices $\mathbf{C}_T$ and $\mathbf{K}_T$ are symmetric, e.g. in elastic conservative and associated plastic systems, all eigenvalues of matrices $(\mathbf{M}^{-1}\cdot\mathbf{C}_T)$ and $(\mathbf{M}^{-1}\cdot\mathbf{K}_T)$ are reals. Inequalities (19) and (20) are thus simplified into

$$\det(\mathbf{C}_T) \geqslant 0 \quad \text{and} \quad \det(\mathbf{K}_T) > 0: \text{necessity for stability}, \tag{21}$$

$$\det(\mathbf{C}_T) < 0 \quad \text{or} \quad \det(\mathbf{K}_T) \leqslant 0: \text{sufficiency for instability}. \tag{22}$$

For subcases of these criteria see Müller (1977).

It should be noted that, although the derivation of the above conditions (21) and (22) is based on the concept of boundedness of perturbed motions, they are not dependent on the perturbed motions, but only dependent on the properties of the discussed system itself.

5. EXAMPLES

We first discuss the simple model of Fig. 1, which is represented by an eccentrically loaded structural system, exhibiting snap-through instabilities. The bar is rigid and of length L, the spring is linear of stiffness k, and the load eccentricity is denoted by e. The bar is assumed to be weightless, and the mass m of the system is concentrated on the top of the rod, point B. Its moment of inertia about the hinge A is denoted by I. The load is applied as a Heaviside function, with constant magnitude P and infinite duration. Thus, the model is a single-degree-of-freedom system and its equation of motion is expressed by Simitses (1989, p. 39):

$$\ddot{\Theta} + \sin\Theta\cdot\cos\Theta - p\cdot(\sin\Theta + \bar{e}\cdot\cos\Theta) = 0, \tag{23}$$

in which the following nondimensionalized parameters are introduced:

$$\bar{e} = e/L; \quad p = (PL)/(ka^2); \quad \tau = ta(k/I)^{1/2}. \tag{24}$$

The perturbed form of eqn (23) is written as

$$\ddot{\tilde{\Theta}} + [\cos(2\Theta) - p(\cos\Theta - \bar{e}\cdot\sin\Theta)]\cdot\tilde{\Theta} = 0. \tag{25}$$

Thus, the sufficient condition for instability (22) for this system leads to

$$p \geqslant \frac{\cos(2\Theta)}{\cos\Theta - \bar{e}\cdot\sin\Theta}. \tag{26}$$

By solving the motion equation (23) numerically, we obtain the responses of the maximum amplitude Θ_{max} versus the nondimensionalized load parameter p, for $e/L = 0.02$. The results are presented graphically on Fig. 2. The instability curve obtained from inequality (26) is also plotted there. We may conclude from both curves that the system will be unstable for $p \geqslant 0.87$, corresponding to Simitses' results (1989). Physically, the motion is simply oscillatory for $p < 0.87$. When $p \geqslant 0.87$ the motion amplifies from a small amplitude ($\Theta \ll \pi/2$) to very large ones ($\Theta > \pi/2$) exhibiting a dynamic snap-through instability.

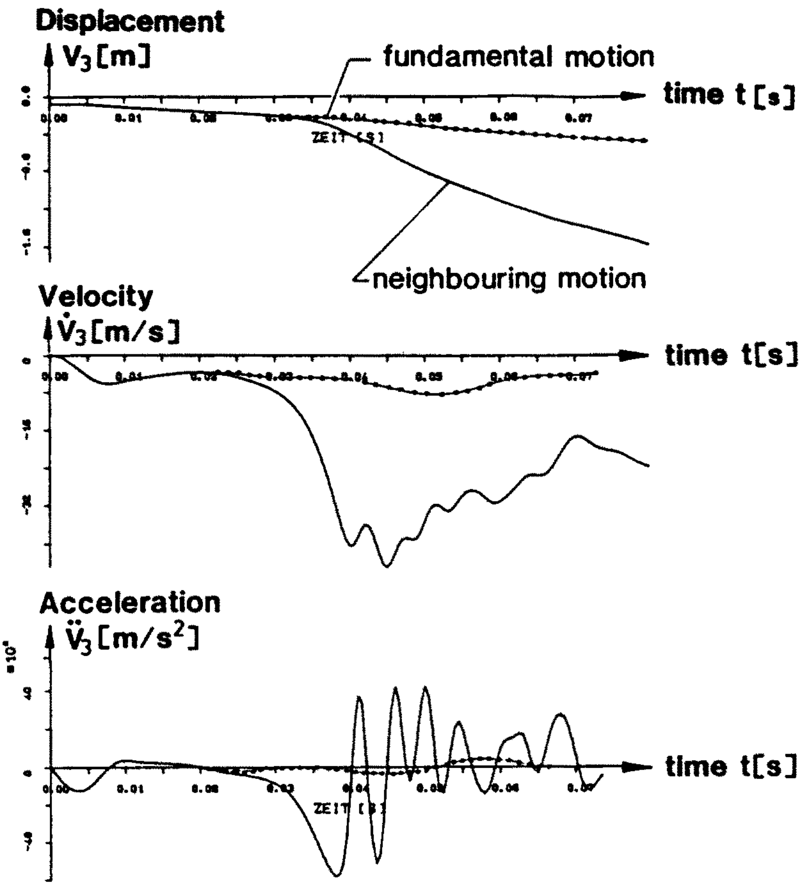

Fig. 4. Time histories for displacement, velocity and acceleration.

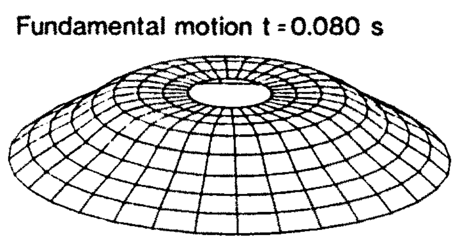

Fig. 5. Graphic display of the deformed shape of spherical cap referred to the fundamental motion ($t = 0.08$ s).

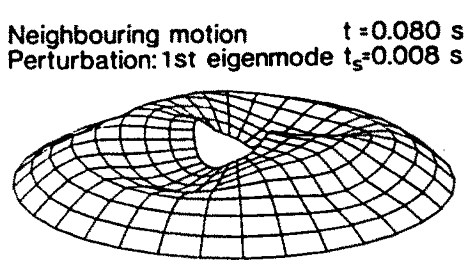

Fig. 6. Graphic display of the deformed shape of spherical cap referred to the perturbed motion ($t = 0.08$ s).

As a second example we select the dynamic instability problem of an isotropic clamped spherical cap with a central opening, subjected to a compound time-dependent load, $\mathbf{P} = \mathbf{P}_L + \mathbf{P}_3$ (Fig. 3). The concentrated load, $\mathbf{P}_L$, along the rim of the opening, and the uniformly distributed load, $\mathbf{P}_3$, are assumed as

$$\mathbf{P}_L = 450 + 225000 \cdot t \,(\mathrm{KN\,m^{-1}}); \quad \mathbf{P}_3 = 1000 + 500000 \cdot t \,(\mathrm{KN\,m^{-2}}). \tag{27}$$

The analysis is accomplished by employing the finite element numerical method based on nonlinear shell formulations (Basar, 1985; Krätzig, 1990; Quante, 1987). The time histories for the displacement, velocity and acceleration are presented in Fig. 4. Because the tangential stiffness matrix $\mathbf{K}_T$ becomes singular at $t = t_{cr} = 0.008$ s, thus we can declare the response is unstable for $t \geqslant t_{cr}$ in terms of our derived sufficient conditions for instability (22). In order to demonstrate our result, the responses of perturbed motion for the displacement, velocity and acceleration at $t = t_{cr}$ are also calculated (see Fig. 4). With reference to these figures, it is clearly observed that the infinitesimal perturbations amplify with time. The graphic displays of the deformed shapes of the isotropic spherical cap with respect to the unperturbed and perturbed motions are shown in Figs 5 and 6, respectively. Obviously, the former represents the deformed shape of axisymmetric fundamental motion, the latter represents the antisymmetric unstable mode.

From the examples discussed above, we may confirm that the dynamic stability problems can be analyzed by the method presented here. The calculations of perturbed motions are avoided.

6. CONCLUSIONS

The dynamic stability of elastic structures has been extensively investigated. There are a number of stability definitions and methods for determining stable or unstable responses. The prevailing definitions and methods follow Liapunov's ideas based on the concept of boundedness of perturbed motions. However, even now it may still be tough to formulate exact critical conditions by using these methods. Despite the fact that it is in principle not difficult to determine the boundedness of a perturbed motion for a particular set of initial conditions, it seems impossible to calculate the responses for all possible initial conditions in the neighborhood of the unperturbed motion. From the viewpoint of the time-saving numerical calculations, it is therefore highly desirous to develop more direct approaches. As in the paper these conditions should be able to determine the stability of dynamical systems in terms of the properties of the system itself and the fundamental state, to avoid the detailed evaluation of perturbed motions for a large number of initial perturbations.

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